

# Lecture 29

Uniform Circuits, TMs with Advice, Karp-Lipton Theorem

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- 2) Accepts when  $x$  is all 1s if and only if advice is 1.

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Let  $L \in \Pi_2^p$ .

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**Proof:** We will prove that if  $SAT \in P_{/poly}$ , then  $\Pi_2^p \subseteq \Sigma_2^p$ . (coC  $\subseteq$  C  $\implies$  C = coC.)

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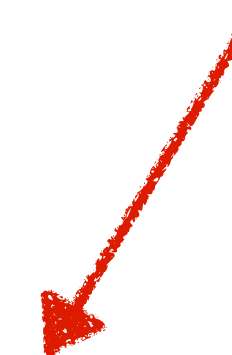
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*Flaw: There might be a circuit  $C$  s.t.  $C(f(x, u_1)) = 1$  even if  $f(x, u_1) \notin \text{SAT}$ .*



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